



Exercices de mathématiques

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Review – Linear Algebra

Exercise 1. 1. Solve the following systems in 4 different ways (by substitution, by the Gauss method, by inverting the matrix of coefficients of the system, by Cramer's formulas):

$$\begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases}$$

2. Choose the method that seems the quickest to you and solve, according to the values of a , the following systems:

$$\begin{cases} ax + y = 2 \\ (a^2 + 1)x + 2ay = 1 \end{cases}$$

$$\begin{cases} (a + 1)x + (a - 1)y = 1 \\ (a - 1)x + (a + 1)y = 1 \end{cases}$$

Correction 1. 1. (a) By substitution

$$\begin{aligned} \begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases} &\Leftrightarrow \begin{cases} 2x + y = 1 \\ x = -\frac{7}{3}y \end{cases} \Leftrightarrow \begin{cases} -\frac{14}{3}y + y = 1 \\ x = -\frac{7}{3}y \end{cases} \\ &\Leftrightarrow \begin{cases} y = -\frac{3}{11} \\ x = \frac{7}{11} \end{cases} \end{aligned}$$

(b) By the Gauss method

$$\begin{aligned} \begin{cases} 2x + y = 1 \\ 3x + 7y = 0 \end{cases} &\Leftrightarrow \begin{cases} 2x + y = 1 \\ 11y = -3 \end{cases} \quad L_2 \leftarrow 2L_2 - 3L_1 \\ &\Leftrightarrow \begin{cases} x = \frac{1-y}{2} \\ y = -\frac{3}{11} \end{cases} \Leftrightarrow \begin{cases} y = -\frac{3}{11} \\ x = \frac{7}{11} \end{cases} \end{aligned}$$

(c) The inverse of the matrix of coefficients of the system is

$$\begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}.$$

Hence the solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{7}{11} \\ -\frac{3}{11} \end{pmatrix}$$

(d) By Cramer's formulas

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = \frac{7}{11} \quad y = \frac{\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = -\frac{3}{11}$$

2. The determinant of the first system is

$$\begin{vmatrix} a & 1 \\ (a^2 + 1) & 2a \end{vmatrix} = a^2 - 1.$$

(a) If $a \notin \{1, -1\}$, one can use Cramer's formulas to obtain:

$$\begin{cases} ax + y = 2 \\ (a^2 + 1)x + 2ay = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2a \end{vmatrix}}{\begin{vmatrix} a & 1 \\ (a^2 + 1) & 2a \end{vmatrix}} = \frac{4a - 1}{a^2 - 1} \\ y = \frac{\begin{vmatrix} a & 2 \\ (a^2 + 1) & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ (a^2 + 1) & 2a \end{vmatrix}} = \frac{-2a^2 + a - 2}{a^2 - 1} \end{cases}$$

(b) If $a = 1$, the system becomes

$$\begin{cases} x + y = 2 \\ 2x + 2y = 1 \end{cases} \Leftrightarrow \begin{cases} x + y = 2 \\ 0 = -1 \quad L_2 \leftarrow L_2 - 2L_1 \end{cases}$$

which is impossible.

(c) If $a = -1$, the system becomes

$$\begin{cases} -x + y = 2 \\ 2x - 2y = 1 \end{cases} \Leftrightarrow \begin{cases} x + y = 2 \\ 0 = 5 \quad L_2 \leftarrow L_2 + 2L_1 \end{cases}$$

which is also impossible.

The determinant of the second system is

$$\begin{vmatrix} (a+1) & (a-1) \\ (a-1) & (a+1) \end{vmatrix} = 4a.$$

(a) If $a \neq 0$, one can use Cramer's formulas to obtain:

$$\begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{\begin{vmatrix} 1 & (a-1) \\ 1 & (a+1) \end{vmatrix}}{4a} = \frac{1}{2a} \\ y = \frac{\begin{vmatrix} (a+1) & 1 \\ (a-1) & 1 \end{vmatrix}}{4a} = \frac{1}{2a} \end{cases}$$

(b) If $a = 0$, the system becomes

$$\begin{cases} x - y = 1 \\ -x + y = 1 \end{cases} \Leftrightarrow \begin{cases} x - y = 1 \\ 0 = 2 \end{cases} \quad L_2 \leftarrow L_2 + L_1$$

which is impossible.

Exercise 2. Solve the following system of 5 equations with 6 unknowns:

$$\begin{cases} 2x + y + z - 2u + 3v - w = 1 \\ 3x + 2y + 2z - 3u + 5v - 3w = 4 \\ 2x + 2y + 2z - 2u + 4v - 4w = 6 \\ x + y + z - u + 2v - 2w = 3 \\ 3x - 3u + 3v + 3w = -6 \end{cases}$$

Correction 2. By the Gauss method

$$\begin{cases} 2x + y + z - 2u + 3v - w = 1 \\ 3x + 2y + 2z - 3u + 5v - 3w = 4 \\ 2x + 2y + 2z - 2u + 4v - 4w = 6 \\ x + y + z - u + 2v - 2w = 3 \\ 3x - 3u + 3v + 3w = -6 \end{cases} \Leftrightarrow \begin{cases} x + y + z - u + 2v - 2w = 3 & L_1 \leftrightarrow L_4 \\ 3x + 2y + 2z - 3u + 5v - 3w = 4 \\ 2x + 2y + 2z - 2u + 4v - 4w = 6 \\ 2x + y + z - 2u + 3v - w = 1 \\ 3x - 3u + 3v + 3w = -6 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z - u + 2v - 2w = 3 \\ -y - z - v + 3w = -5 & L_2 \leftarrow L_2 - 3L_1 \\ 0 = 0 & L_3 \leftarrow L_3 - 2L_1 \\ -y - z - v + 3w = -5 & L_4 \leftarrow L_4 - 2L_1 \\ -3y - 3z - 3v + 9w = -15 & L_5 \leftarrow L_5 - 3L_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y + z - u + 2v - 2w = 3 \\ -y - z - v + 3w = -5 \end{cases}$$

It follows that the set of solutions is a 4-space in \mathbb{R}^6 . Let us parametrize the set of solutions by $a = z \in \mathbb{R}$, $b = u \in \mathbb{R}$, $c = v \in \mathbb{R}$, $d = w \in \mathbb{R}$. One obtains

$$\begin{cases} x = -y - a + b - 2c + 2d + 3 = b - c - d - 2 \\ y = -a - c + 3d + 5 \\ z = a \\ u = b \\ v = c \\ w = d \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 3. For each pair (A_i, b_i) , $1 \leq i \leq 5$ of matrices below

1. give the nature of the set of solutions of the system $A_i X = b_i$;
2. give a parametric representation of the set of solutions of $A_i X = b_i$;
3. give a basis of the range and a basis of the kernel of A_i .

$$\begin{aligned}
\text{a) } A_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & b_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; & \text{b) } A_2 &= \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} & b_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \\
\text{c) } A_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & b_3 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; & \text{d) } A_4 &= \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & b_4 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \\
\text{e) } A_5 &= \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & b_5 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix};
\end{aligned}$$

Correction 3. a) Since $\det A_1 = 1 \neq 0$, the matrix A_1 is invertible hence defines an isomorphism of \mathbb{R}^4 . The system $A_1 X = b_1$ has therefore a unique solution given by $X = A_1^{-1} b_1 = (0, 0, -1, 1)^T$ by a standard computation. The range of A_1 is \mathbb{R}^4 , hence the canonical basis of \mathbb{R}^4 is a basis of $\text{Im } A_1$. The kernel of A_1 is $\{\vec{0}\}$, hence a basis of $\ker A_1$ is \emptyset .

b) The rank of A_2 is 4, hence the dimension of the kernel of A_2 is 1. Therefore the set of solutions of $A_2 X = b_2$ is an affine line in \mathbb{R}^5 parallel to $\ker A_2$. Denote by (x, y, z, t, u) the coordinates in \mathbb{R}^5 . Let us parametrize the set of solutions by $a = u \in \mathbb{R}$. The system is equivalent to

$$\begin{aligned}
& \begin{cases} x + 2y + t = 1 - 3a \\ y + z + t = 1 - 2a \\ z + 2t = 1 - 3a \\ t = 1 - a \end{cases} \Leftrightarrow \begin{cases} x = 1 - 3a - 2y - t \\ y = 1 - 2a - z - t \\ z = 1 - 3a - 2t \\ t = 1 - a \end{cases} \\
& \Leftrightarrow \begin{cases} x = 1 - 3a - 2 - 1 + a = -2 - 2a \\ y = 1 - 2a + 1 + a + -(1 - a) = 1 \\ z = 1 - 3a - 2 + 2a = -1 - a \\ t = 1 - a \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, a \in \mathbb{R}.
\end{aligned}$$

Since A_2 is surjective, the canonical basis of \mathbb{R}^4 is a basis of $\text{Im } A_2$. The previous resolution implies that a basis of $\ker A_2$ is given by the single

$$\text{vector } \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

c) Since the last equation of the system is impossible, the system $A_3X = b_3$ admits no solution. The rank of A_3 is 4, therefore by the Rank theorem, the dimension of $\ker A_3$ is 0. A basis of $\text{Im}A_3$ is given by the 4 columns of A_3 . A basis of $\ker A_3$ is given by the empty set \emptyset .

d) The last equation of $A_4X = b_4$ is impossible, hence this system admits no solution. The rank of A_4 is 4, hence by the Rank theorem, the dimension of the kernel of A_4 is 1. A basis of $\text{Im}A_4$ is given by the first 4 columns of A_4 . A basis of $\ker A_4$ is a nontrivial vector $X \in \mathbb{R}^5$

solution of $A_4X = \vec{0}$. One finds that $\begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ generates $\ker A_4$.

e) For the basis of $\text{Im}A_5$ and $\ker A_5$ see d). The vector b_5 belongs to $\text{Im}A_5$ since the last equation (compatibility condition) is satisfied. The kernel of A_5 being a line, the set of solutions of $A_5X = b_5$ is an affine line in

\mathbb{R}^5 parallel to $\ker A_5$. Since the vector $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is a particular solution

of the system, one obtains that the set of solutions is parametrized by

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, a \in \mathbb{R}.$$

Exercise 4. Compute a basis of the image and a basis of the kernel of the linear application

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^5 \\ (x, y, z) \longmapsto (x + y, x + y + z, 2x + y + z, 2x + 2y + z, y + z)$$

What is the rank of f ?

Correction 4. The matrix of the linear application f is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let us compute a basis of $\text{Im} f$ and a basis of $\ker f$. One has:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Consequently the kernel of f is trivial, and a basis of $\text{Im} f$ is given by $v_1 =$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \text{ The rank of } f \text{ is the dimension of}$$

$\text{Im} f$, that is, 3.

Exercise 5. Let A be the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

1. Consider the matrices $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

Show that $AB = AC$. Can the matrix A be invertible?

2. Determine all matrices F of size $(3,3)$ such that $AF = 0$ (where 0 denotes the matrix all of whose entries are zero).

Correction 5. 1. One has

$$AB = AC = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}.$$

Suppose that the matrix A is invertible. Multiply both members of the equation $AB = AC$ on the left by A^{-1} to get $B = C$. But the matrices B and C are not equal. This is a contradiction. Hence the matrix A is not invertible.

2. Let F be any real matrix $(3,3)$

$$F = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

The equation $AF = 0$ gives rise to the following system

$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d + g = 0 \\ e + h = 0 \\ f + i = 0 \\ 3a + d + g = 0 \\ 3b + e + h = 0 \\ 3c + f + i = 0 \end{cases}$$

Consequently the set of matrices F such that $AF = 0$ is the set of matrices of the form

$$F = \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ -d & -e & -f \end{pmatrix}, d \in \mathbb{R}, e \in \mathbb{R}, f \in \mathbb{R}.$$

Exercise 6. For which values of a is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{pmatrix}$$

invertible? Compute in this case its inverse.

Correction 6. One has

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & a \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2a - 12 - (a - 3) + 2 = a - 7.$$

Hence A is invertible if and only if $a \neq 7$. In this case, the standard algorithm yields

$$A^{-1} = \frac{1}{a-7} \begin{pmatrix} 2a-12 & 3-a & 2 \\ 4-a & a-1 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Exercise 7. Let a and b be two real numbers, and A be the matrix

$$A = \begin{pmatrix} a & 2 & -1 & b \\ 3 & 0 & 1 & -4 \\ 5 & 4 & -1 & 2 \end{pmatrix}$$

Show that $\text{rk}(A) \geq 2$ (where rk denotes the rank). For which values of a and b is the rank of A equal to 2?

Correction 7. Recall that the rank of A is the greatest number of columns of A that are linearly independent. Since the second and third columns C_2, C_3 of A are not proportional, they are linearly independent. Therefore the rank of A is at least 2. For the rank of A to be exactly 2, one has to impose that the first and last columns of A are each a linear combination of C_2 and C_3 (which are fixed). The only linear combination of C_2 and C_3 that has the form $(a, 3, 5)^T$ is $3C_3 + 2C_2 = (1, 3, 5)^T$, hence $a = 1$. The only linear combination of C_2 and C_3 that has the form $(b, -4, 2)^T$ is $-4C_3 - \frac{1}{2}C_2 = (3, -4, 2)^T$, hence $b = 3$. Consequently the rank of A is 2 if and only if $a = 1$ and $b = 3$.

Exercise 8. Compute the inverse of the following matrix

$$A = \begin{pmatrix} 4 & 8 & 7 & 4 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Correction 8. One obtains

$$A^{-1} = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 4 & -4 \\ 1 & 0 & -4 & 5 \end{pmatrix}.$$

Exercise 9. Let us denote by $\{e_1, e_2, \dots, e_n\}$ the canonical basis of \mathbb{R}^n . To a permutation $\sigma \in \mathcal{S}_n$, one associates the following endomorphism u_σ of \mathbb{R}^n :

$$u_\sigma : \begin{matrix} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \longmapsto & \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix} \end{matrix}$$

1. Let $\tau = (ij)$ be a transposition. Write the matrix of u_τ in the canonical basis. Show that $\det(u_\tau) = -1$.
2. Show that $\forall \sigma, \sigma' \in \mathcal{S}_n, u_\sigma \circ u_{\sigma'} = u_{\sigma' \circ \sigma}$.
3. Show that $\forall \sigma \in \mathcal{S}_n, \det u_\sigma = \varepsilon(\sigma)$ where ε denotes the signature.

Correction 9. 1. Let τ be the transposition which exchanges i and j .

homomorphism from \mathcal{S}_n into $\{\pm 1\}$. Since the transpositions generate the group of permutations \mathcal{S}_n , two group homomorphisms from \mathcal{S}_n to $\{\pm 1\}$ which coincide on the set of transpositions coincide on \mathcal{S}_n . By 1., the group homomorphism from \mathcal{S}_n into $\{\pm 1\}$ which maps σ onto $\det u_{\sigma^{-1}}$ coincides with the signature on the set of transpositions, because a transposition is its own inverse. Hence $\forall \sigma \in \mathcal{S}_n$, $\det u_{\sigma} = \varepsilon(\sigma)$.

Exercise 10. 1. Compute the eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 2 \\ 1 & -3 & 4 \end{pmatrix}.$$

2. Compute A^n for all $n \in \mathbb{N}$.

Correction 10. 1. One has

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 2 & -2 \\ 1 & -1 - \lambda & 2 \\ 1 & -3 & 4 - \lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -1 - \lambda & 2 \\ -3 & 4 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & -2 \\ -3 & 4 - \lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & -2 \\ -1 - \lambda & 2 \end{vmatrix} \\ &= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda - 2)(\lambda - 1). \end{aligned}$$

Therefore the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 1$.

A nontrivial vector in the kernel of A is given by $v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Let us find a vector generating the eigenspace associated to $\lambda_2 = 2$. One has

$$\begin{pmatrix} A - \lambda_2 I \\ I \end{pmatrix} = \begin{pmatrix} -2 & 2 & -2 \\ 1 & -3 & 2 \\ 1 & -3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{C_2 \leftarrow C_2 + C_1 \\ C_3 \leftarrow C_3 - C_1}} \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3 \leftarrow 2C_3 + C_2} \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

It follows that the vector $v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is a basis of the eigenspace associated to $\lambda_2 = 2$. Now one has

$$A - I = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 2 \\ 1 & -3 & 3 \end{pmatrix}.$$

Consequently the vector $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ generates the eigenspace associated to $\lambda_3 = 1$.

2. Denote by f the linear application whose matrix in the canonical basis of \mathbb{R}^3 is A . The vectors v_1, v_2 and v_3 form a basis of \mathbb{R}^3 . In this new basis, the matrix of f is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The relation between A and D is $D = P^{-1}AP$ where $P = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

The inverse of P is

$$P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Therefore, for $n > 0$, we have $A^n = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$: cancelling all occurrences of $P^{-1}P = I$ one gets

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1^n \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2^n & -2^n \\ 1 & -2^n + 1 & 2^n \\ 1 & -2^{n+1} + 1 & 2^{n+1} \end{pmatrix}. \end{aligned}$$