Exercices de mathématiques

Exercises and corrections: Barbara Tumpach

Review – Linear Algebra

Exercise 1. 1. Solve the following systems in 4 different ways (by substitution, by the Gauss method, by inverting the matrix of coefficients of the system, by Cramer’s formulas):

\[
\begin{aligned}
2x + y &= 1 \\
3x + 7y &= 0
\end{aligned}
\]

2. Choose the method that seems the quickest to you and solve, according to the values of \(a\), the following systems:

\[
\begin{aligned}
ax + y &= 2 \\
(a^2 + 1)x + 2ay &= 1
\end{aligned}
\]

Correction 1. 1. (a) By substitution

\[
\begin{aligned}
2x + y &= 1 \\ 3x + 7y &= 0
\end{aligned} \iff \begin{aligned}
2x + y &= 1 \\
&= \frac{1}{3}
\end{aligned} \iff \begin{aligned}
-\frac{14}{3}y + y &= 1 \\
&= \frac{1}{3} \\
x &= \frac{-7}{3} \iff \begin{aligned}
y &= \frac{-3}{11} \\
x &= \frac{7}{11}
\end{aligned}
\]

(b) By the Gauss method

\[
\begin{aligned}
2x + y &= 1 \\ 3x + 7y &= 0
\end{aligned} \iff \begin{aligned}
2x + y &= 1 \\
11y &= -3
\end{aligned} \iff \begin{aligned}
L_2 &\leftarrow 2L_2 - 3L_1 \\
x &= \frac{1-y}{3} \\
y &= \frac{-3}{11}
\end{aligned} \iff \begin{aligned}
y &= \frac{-3}{11} \\
x &= \frac{3}{11}
\end{aligned}
\]
(c) The inverse of the matrix of coefficients of the system is
\[
\begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}.
\]
Hence the solution of the system is
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left( \frac{7}{11}, \frac{-3}{11} \right)
\]

(d) By Cramer’s formulas
\[
x = \frac{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}}{\begin{pmatrix} 0 & 7 \\ 3 & 7 \end{pmatrix}} = \frac{7}{11}, \quad y = \frac{\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}}{\begin{pmatrix} 0 & 7 \\ 3 & 7 \end{pmatrix}} = \frac{-3}{11}
\]

2. The determinant of the first system is
\[
\left| \begin{array}{cc} a & 1 \\ (a^2 + 1) & 2a \end{array} \right| = a^2 - 1.
\]
(a) If \(a \notin \{1, -1\}\), one can use Cramer’s formulas to obtain:
\[
\begin{cases} \frac{a}{(a^2 + 1)} \begin{vmatrix} 2 & 1 \\ a & 1 \end{vmatrix} = \frac{4a - 1}{a^2 - 1} \\
\frac{1}{(a^2 + 1)2a} \begin{vmatrix} a & 2 \\ a & 1 \end{vmatrix} = \frac{-2a^2 + a - 2}{a^2 - 1}
\end{cases}
\]
(b) If \(a = 1\), the system becomes
\[
\begin{cases} \frac{x + y = 2}{2x + 2y = 1} \iff \begin{cases} x + y = 2 \\ 0 = -1 \end{cases} \quad \leftarrow L_2 \leftarrow L_2 - 2L_1
\end{cases}
\]
which is impossible.
(c) If \(a = -1\), the system becomes
\[
\begin{cases} \frac{-x + y = 2}{2x - 2y = 1} \iff \begin{cases} x + y = 2 \\ 0 = 5 \end{cases} \quad \leftarrow L_2 \leftarrow L_2 + 2L_1
\end{cases}
\]
which is also impossible.
The determinant of the second system is
\[
\begin{vmatrix}
(a+1) & (a-1) \\
(a-1) & (a+1)
\end{vmatrix} = 4a.
\]

(a) If \(a \neq 0\), one can use Cramer’s formulas to obtain:
\[
\begin{aligned}
\begin{cases}
(a+1)x + (a-1)y = 1 \\
(a-1)x + (a+1)y = 1
\end{cases}
\iff
\begin{cases}
x = \frac{1}{4a} \begin{vmatrix}
1 & (a-1) \\
1 & (a+1)
\end{vmatrix} = \frac{1}{2a} \\
y = \frac{1}{4a} \begin{vmatrix}
(a+1) & 1 \\
(a-1) & 1
\end{vmatrix} = \frac{1}{2a}
\end{cases}
\end{aligned}
\]

(b) If \(a = 0\), the system becomes
\[
\begin{aligned}
\begin{cases}
x - y = 1 \\
x + y = 1
\end{cases}
\iff
\begin{cases}
x - y = 1 \\
x = 2 \ L_2 \leftrightarrow L_2 + L_1
\end{cases}
\end{aligned}
\]
which is impossible.

Exercise 2. Solve the following system of 5 equations with 6 unknowns:
\[
\begin{aligned}
\begin{cases}
2x + y + z - 2u + 3v - w = 1 \\
3x + 2y + 2z - 3u + 5v - 3w = 4 \\
2x + 2y + 2z - 2u + 4v - 4w = 6 \\
x + y + z - u + 2v - 2w = 3 \\
3x - 3u + 3v + 3w = -6
\end{cases}
\end{aligned}
\]

Correction 2. By the Gauss method
\[
\begin{aligned}
\begin{cases}
2x + y + z - 2u + 3v - w = 1 \\
3x + 2y + 2z - 3u + 5v - 3w = 4 \\
2x + 2y + 2z - 2u + 4v - 4w = 6 \\
x + y + z - u + 2v - 2w = 3 \\
3x - 3u + 3v + 3w = -6
\end{cases}
\iff
\begin{cases}
x + y + z - u + 2v - 2w = 3 \ L_1 \leftrightarrow L_4 \\
3x + 2y + 2z - 3u + 5v - 3w = 4 \\
2x + 2y + 2z - 2u + 4v - 4w = 6 \\
2x + y + z - 2u + 3v - w = 1 \\
3x - 3u + 3v + 3w = -6
\end{cases}
\end{aligned}
\]
\[\begin{align*}
&\Rightarrow \left\{ \begin{array}{l}
x + y + z - u + 2v - 2w = 3 \\
- y - z - v + 3w = -5 \quad L_2 \leftrightarrow L_2 - 3L_1 \\
- y - z - v + 3w = -5 \quad L_3 \leftrightarrow L_3 - 2L_1 \\
- 3y - 3z - 3v + 9w = -15 \quad L_4 \leftrightarrow L_4 - 2L_1 \\
\end{array} \right. \\
&\Rightarrow \left\{ \begin{array}{l}
x + y + z - u + 2v - 2w = 3 \\
- y - z - v + 3w = -5 \\
\end{array} \right.
\end{align*}\]

It follows that the set of solutions is a 4-space in \(\mathbb{R}^6\). Let us parametrize the set of solutions by \(a = z \in \mathbb{R}, b = u \in \mathbb{R}, c = v \in \mathbb{R}, d = w \in \mathbb{R}\). One obtains

\[\begin{align*}
&x = -y - a + b - 2c + 2d + 3 = b - c - d - 2 \\
y = -a - c + 3d + 5 \\
z = a \\
u = b \\
v = c \\
w = d
\end{align*}\]

\[\begin{pmatrix}
x \\
y \\
z \\
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix}
-2 \\
5 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + a \begin{pmatrix}
0 \\
-1 \\
1 \\
0 \\
0 \\
0
\end{pmatrix} + b \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + c \begin{pmatrix}
1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} + d \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

**Exercise 3.** For each pair \((A_i, b_i), 1 \leq i \leq 5\) of matrices below

1. give the nature of the set of solutions of the system \(A_iX = b_i\);

2. give a parametric representation of the set of solutions of \(A_iX = b_i\);

3. give a basis of the range and a basis of the kernel of \(A_i\).
a) \( A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) \( b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \); b) \( A_2 = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \) \( b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \);

b) The rank of \( A_2 \) is 4, hence the dimension of the kernel of \( A_2 \) is 1. Therefore the set of solutions of \( A_2X = b_2 \) is an affine line in \( \mathbb{R}^5 \) parallel to \( \ker A_2 \). Denote by \((x, y, z, t, u)\) the coordinates in \( \mathbb{R}^5 \). Let us parametrize the set of solutions by \( a = u \in \mathbb{R} \). The system is equivalent to

\[
\begin{align*}
    x + 2y + t &= 1 - 3a \\
    y + z + t &= 1 - 2a \\
    z + 2t &= 1 - 3a \\
    t &= 1 - a
\end{align*}
\]

\[
\begin{align*}
    x &= 1 - 3a - 2y - t \\
    y &= 1 - 2a - z - t \\
    z &= 1 - 3a - 2t \\
    t &= 1 - a
\end{align*}
\]

\[
\begin{align*}
    x &= -2 \\
    y &= 1 \\
    z &= -1 \\
    t &= 1 - a
\end{align*}
\]

Since \( A_2 \) is surjective, the canonical basis of \( \mathbb{R}^4 \) is a basis of \( \text{Im} A_2 \). The previous resolution implies that a basis of \( \ker A_2 \) is given by the single vector

\[
\begin{pmatrix}
    -2 \\
    0 \\
    -1 \\
    1
\end{pmatrix}
\]
c) Since the last equation of the system is impossible, the system \( A_3 X = b_3 \) admits no solution. The rank of \( A_3 \) is 4, therefore by the Rank theorem, the dimension of \( \ker A_3 \) is 0. A basis of \( \text{Im} A_3 \) is given by the 4 columns of \( A_3 \). A basis of \( \ker A_3 \) is given by the empty set \( \emptyset \).

d) The last equation of \( A_4 X = b_4 \) is impossible, hence this system admits no solution. The rank of \( A_4 \) is 4, hence by the Rank theorem, the dimension of the kernel of \( A_4 \) is 1. A basis of \( \text{Im} A_4 \) is given by the first 4 columns of \( A_4 \). A basis of \( \ker A_4 \) is a nontrivial vector \( X \in \mathbb{R}^5 \) solution of \( A_4 X = \vec{0} \). One finds that

\[
\begin{pmatrix}
2 \\
-1 \\
1 \\
-1 \\
1
\end{pmatrix}
\]

generates \( \ker A_4 \).

e) For the basis of \( \text{Im} A_5 \) and \( \ker A_5 \) see d). The vector \( b_5 \) belongs to \( \text{Im} A_5 \) since the last equation (compatibility condition) is satisfied. The kernel of \( A_5 \) being a line, the set of solutions of \( A_5 X = b_5 \) is an affine line in \( \mathbb{R}^5 \) parallel to \( \ker A_5 \). Since the vector

\[
\begin{pmatrix}
2 \\
-1 \\
0 \\
0 \\
1
\end{pmatrix}
\]

is a particular solution of the system, one obtains that the set of solutions is parametrized by

\[
\begin{pmatrix}
x \\
y \\
z \\
t \\
u
\end{pmatrix} = \begin{pmatrix}
2 \\
-1 \\
1 \\
0 \\
1
\end{pmatrix} + a \begin{pmatrix}
2 \\
-1 \\
1 \\
0 \\
1
\end{pmatrix}, \ a \in \mathbb{R}.
\]

**Exercise 4.** Compute a basis of the image and a basis of the kernel of the linear application

\[
f : \mathbb{R}^3 \longrightarrow \mathbb{R}^5 \\
(x, y, z) \longmapsto (x + y, x + y + z, 2x + y + z, 2x + 2y + z, y + z)
\]

What is the rank of \( f \)?

**Correction 4.** The matrix of the linear application \( f \) is

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
2 & 1 & 1 \\
2 & 2 & 1 \\
0 & 1 & 1
\end{pmatrix}.
\]
Let us compute a basis of $\text{Im} f$ and a basis of $\ker f$. One has:

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
2 & 1 & 1 \\
2 & 2 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 1 \\
2 & -1 & 1 \\
2 & 0 & 1 \\
0 & 1 & 1 \\
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
2 & 1 & -1 \\
2 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

Consequently the kernel of $f$ is trivial, and a basis of $\text{Im} f$ is given by $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. The rank of $f$ is the dimension of $\text{Im} f$, that is, 3.

**Exercise 5.** Let $A$ be the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
3 & 1 & 1
\end{pmatrix}
\]

1. Consider the matrices $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

Show that $AB = AC$. Can the matrix $A$ be invertible?

2. Determine all matrices $F$ of size $(3, 3)$ such that $AF = 0$ (where 0 denotes the matrix all of whose entries are zero).

**Correction 5.**

1. One has

\[
AB = AC = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}
\]

Suppose that the matrix $A$ is invertible. Multiply both members of the equation $AB = AC$ on the left by $A^{-1}$ to get $B = C$. But the matrices $B$ and $C$ are not equal. This is a contradiction. Hence the matrix $A$ is not invertible.

2. Let $F$ be any real matrix $(3, 3)$

\[
F = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}
\]

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The equation $AF = 0$ gives rise to the following system

$$
\begin{align*}
& a = 0 \\
& b = 0 \\
& c = 0 \\
& d + g = 0 \\
& e + h = 0 \\
& f + i = 0 \\
& 3a + d + g = 0 \\
& 3b + e + h = 0 \\
& 3c + f + i = 0
\end{align*}
$$

Consequently the set of matrices $F$ such that $AF = 0$ is the set of matrices of the form

$$F = \begin{pmatrix} 0 & 0 & 0 \\ d & e & f \\ -d & -e & -f \end{pmatrix}, d \in \mathbb{R}, e \in \mathbb{R}, f \in \mathbb{R}.$$

**Exercise 6.** For which values of $a$ is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{pmatrix}$$

invertible? Compute in this case its inverse.

**Correction 6.** One has

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{vmatrix} = 2a - 12 - (a - 3) + 2 = a - 7.$$

Hence $A$ is invertible if and only if $a \neq 7$. In this case, the standard algorithm yields

$$A^{-1} = \frac{1}{a - 7} \begin{pmatrix} 2a - 12 & 3 - a & 2 \\ 4 - a & a - 1 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

**Exercise 7.** Let $a$ and $b$ be two real numbers, and $A$ be the matrix

$$A = \begin{pmatrix} a & 2 & -1 & b \\ 3 & 0 & 1 & -4 \\ 5 & 4 & -1 & 2 \end{pmatrix}$$

Show that $\text{rk}(A) \geq 2$ (where $\text{rk}$ denotes the rank). For which values of $a$ and $b$ is the rank of $A$ equal to 2?
Correction 7. Recall that the rank of $A$ is the greatest number of columns of $A$ that are linearly independent. Since the second and third columns $C_2, C_3$ of $A$ are not proportional, they are linearly independent. Therefore the rank of $A$ is at least 2. For the rank of $A$ to be exactly 2, one has to impose that the first and last columns of $A$ are each a linear combination of $C_2$ and $C_3$ (which are fixed). The only linear combination of $C_2$ and $C_3$ that has the form $(a, 3, 5)^T$ is $3C_3 + 2C_2 = (1, 3, 5)^T$, hence $a = 1$. The only linear combination of $C_2$ and $C_3$ that has the form $(b, -4, 2)^T$ is $-4C_3 - \frac{1}{2}C_2 = (3, -4, 2)^T$, hence $b = 3$. Consequently the rank of $A$ is 2 if and only if $a = 1$ and $b = 3$.

Exercise 8. Compute the inverse of the following matrix

$$A = \begin{pmatrix} 4 & 8 & 7 & 4 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Correction 8. One obtains

$$A^{-1} = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 4 & -4 \\ 1 & 0 & -4 & 5 \end{pmatrix}.$$

Exercise 9. Let us denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. To a permutation $\sigma \in S_n$, one associates the following endomorphism $u_{\sigma}$ of $\mathbb{R}^n$:

$$u_{\sigma} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(n)} \end{pmatrix}$$

1. Let $\tau = (ij)$ be a transposition. Write the matrix of $u_{\tau}$ in the canonical basis. Show that $\det(u_{\tau}) = -1$.

2. Show that $\forall \sigma, \sigma' \in S_n, \ u_{\sigma} \circ u_{\sigma'} = u_{\sigma' \circ \sigma}$.

3. Show that $\forall \sigma \in S_n, \det u_{\sigma} = \varepsilon(\sigma)$ where $\varepsilon$ denotes the signature.

Correction 9. 1. Let $\tau$ be the transposition which exchanges $i$ and $j$. 

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The matrix of $u_r$ in the canonical basis of $\mathbb{R}^n$ is

$$
\begin{pmatrix}
i & \ldots & j \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \mathbf{0} & 0 & \ldots & \mathbf{1} & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}
$$

By exchanging the columns $i$ and $j$ of the matrix of $u_r$ one obtains the identity matrix. Therefore $\det u_r = -\det I = -1$, where $I$ denotes the identity matrix.

2. For any $\sigma, \sigma' \in S_n$, one has

$$
u_{\sigma} \circ u_{\sigma'} \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) = u_{\sigma} \left( \begin{array}{c} x_{\sigma'(1)} \\ \vdots \\ x_{\sigma'(n)} \end{array} \right) = \left( \begin{array}{c} x_{\sigma'(\sigma(1))} \\ \vdots \\ x_{\sigma'(\sigma(n))} \end{array} \right) = u_{\sigma' \circ \sigma} \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)$$

Since the previous equality is satisfied for every $\left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)$ in $\mathbb{R}^n$, it implies that $u_{\sigma} \circ u_{\sigma'} = u_{\sigma' \circ \sigma}$.

An alternative proof is to check that $u_{\sigma}$ sends each $e_i$ to $e_{\sigma^{-1}(i)}$ (the basis vector whose only nonzero coordinate is the $\sigma^{-1}(i)$-th): hence,

$$u_{\sigma} \circ u_{\sigma'}(e_i) = u_{\sigma}(e_{\sigma'^{-1}(i)}) = e_{\sigma^{-1}(\sigma'^{-1}(i))} = e_{(\sigma' \circ \sigma)^{-1}(i)} = u_{\sigma' \circ \sigma}(i).$$

3. By 2., the map which associates $u_{\sigma^{-1}}$ to a permutation $\sigma$ is a group homomorphism from $S_n$ into the group of invertible matrices of size $(n, n)$, because $u_{\sigma^{-1}} \circ u_{\sigma^{-1}} = u_{\sigma^{-1} \circ \sigma^{-1}} = u_{(\sigma \circ \sigma')^{-1}}$. Consequently, the map which assigns to a permutation $\sigma$ the number $\det u_{\sigma^{-1}}$ is a group
homomorphism from $S_n$ into $\{\pm 1\}$. Since the transpositions generate the group of permutations $S_n$, two group homomorphisms from $S_n$ to $\{\pm 1\}$ which coincide on the set of transpositions coincide on $S_n$. By 1., the group homomorphism from $S_n$ into $\{\pm 1\}$ which maps $\sigma$ onto $\det u_{\sigma^{-1}}$ coincides with the signature on the set of transpositions, because a transposition is its own inverse. Hence $\forall \sigma \in S_n$, $\det u_{\sigma} = \varepsilon(\sigma)$.

**Exercise 10.**  1. Compute the eigenvalues and eigenvectors of the following matrix

\[
A = \begin{pmatrix}
0 & 2 & -2 \\
1 & -1 & 2 \\
1 & -3 & 4
\end{pmatrix}.
\]

2. Compute $A^n$ for all $n \in \mathbb{N}$.

**Correction 10.**  1. One has

\[
\det (A - \lambda I) = \begin{vmatrix}
-\lambda & 2 & -2 \\
1 & -1 - \lambda & 2 \\
1 & -3 & 4 - \lambda
\end{vmatrix}
\]

\[
= -\lambda \begin{vmatrix}
-1 - \lambda & 2 \\
-3 & 4 - \lambda
\end{vmatrix} - 1 \begin{vmatrix}
2 & -2 \\
-3 & 4 - \lambda
\end{vmatrix} + 1 \begin{vmatrix}
2 & -2 \\
-1 - \lambda & 2
\end{vmatrix}
\]

\[
= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda (\lambda - 2) (\lambda - 1).
\]

Therefore the eigenvalues of $A$ are $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 1$.

A nontrivial vector in the kernel of $A$ is given by $v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Let us find a vector generating the eigenspace associated to $\lambda_2 = 2$. One has

\[
\begin{pmatrix}
-2 & 2 & -2 \\
1 & -3 & 2 \\
1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-2 & 0 & 0 \\
1 & -2 & 1 \\
1 & -2 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-2 & 0 & 0 \\
1 & -2 & 0 \\
1 & -2 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -1 \\
0 & 1 & 1 \\
0 & 0 & 2
\end{pmatrix}
\]
It follows that the vector $v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is a basis of the eigenspace associated to $\lambda_2 = 2$. Now one has

$$A - I = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 2 \\ 1 & -3 & 3 \end{pmatrix}.$$ 

Consequently the vector $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ generates the eigenspace associated to $\lambda_3 = 1$.

2. Denote by $f$ the linear application whose matrix in the canonical basis of $\mathbb{R}^3$ is $A$. The vectors $v_1, v_2$ and $v_3$ form a basis of $\mathbb{R}^3$. In this new basis, the matrix of $f$ is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The relation between $A$ and $D$ is $D = P^{-1}AP$ where $P = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$. The inverse of $P$ is

$$P^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$ 

Therefore, for $n > 0$, we have $A^n = (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})$: cancelling all occurrences of $P^{-1}P = I$ one gets

$$A^n = PD^nP^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1^n \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2^n & -2^n \\ 1 & -2^n + 1 & 2^n \\ 1 & 2^{n+1} + 1 & 2^{n+1} \end{pmatrix}.$$ 

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