Exercices de mathématiques

Exercises and corrections: Barbara Tumpach

**Linear maps — subvectorspaces of \( \mathbb{R}^n \)**

**Exercise 1.** 1. Endow \( \mathbb{R}^2 \) with an orthonormal frame \((O, \vec{i}, \vec{j})\). Show that a linear map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) is uniquely determined by its values on the vectors \( \vec{i} \) and \( \vec{j} \).

2. In the basis \( \{\vec{i}, \vec{j}\} \), what is the matrix of the orthogonal symmetry with respect to the horizontal axis?

3. In the basis \( \{\vec{i}, \vec{j}\} \), what is the matrix of the orthogonal projection to the horizontal axis?

4. In the basis \( \{\vec{i}, \vec{j}\} \), what is the matrix of the rotation of angle \( \theta \) and center \( O \)?

5. In the basis \( \{\vec{i}, \vec{j}\} \), what is the matrix of the homothety of center \( O \) and ratio \( k \)?

6. In the basis \( \{\vec{i}, \vec{j}\} \), what is the matrix of the symmetry of center \( O \)?

7. Is a translation a linear map?

**Correction 1.** 1. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear map. Consider any vector \( \vec{v} \) in \( \mathbb{R}^2 \). Since \( \{\vec{i}, \vec{j}\} \) is a basis of \( \mathbb{R}^2 \), \( \vec{v} \) can be uniquely written as: \( \vec{v} = x\vec{i} + y\vec{j} \). By linearity of \( f \), one has: \( f(\vec{v}) = f(x\vec{i} + y\vec{j}) = xf(\vec{i}) + yf(\vec{j}) \). Therefore the values of \( f \) on the vectors \( \vec{i} \) and \( \vec{j} \), determine the value of \( f \) on any vector of \( \mathbb{R}^2 \). Two linear maps taking the same values on \( \vec{i} \) and \( \vec{j} \) will coincide on \( \mathbb{R}^2 \).

2. In the basis \( \{\vec{i}, \vec{j}\} \), the matrix of the orthogonal symmetry with respect to the horizontal axis is \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

3. In the basis \( \{\vec{i}, \vec{j}\} \), the matrix of the orthogonal projection to the horizontal axis is \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).
4. In the basis \( \{ \vec{i}, \vec{j} \} \), the matrix of the rotation of angle \( \theta \) and center \( O \) is 
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

5. In the basis \( \{ \vec{i}, \vec{j} \} \), the matrix of the homothety of center \( O \) and ratio \( k \) is 
\[
\begin{pmatrix}
k & 0 \\
0 & k
\end{pmatrix}.
\]

6. In the basis \( \{ \vec{i}, \vec{j} \} \), the matrix of the symmetry of center \( O \) is 
\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}.
\]

7. A linear map \( f \) from \( \mathbb{R}^n \) into \( \mathbb{R}^p \) necessarily maps \( \vec{0} \in \mathbb{R}^n \) onto \( \vec{0} \in \mathbb{R}^p \).

Exercise 2. Let \( f \) be the map from \( \mathbb{R}^4 \) to \( \mathbb{R}^4 \) defined by:
\[
f(x, y, z, t) = (x + y + z + t, x + y + z + t, x + y + z + t, 2x + 2y + 2z + 2t).
\]

1. Show that \( f \) is linear and write down its matrix in the canonical basis of \( \mathbb{R}^4 \).

2. Check that the vectors \( \vec{a} = (1, -1, 0, 0) \), \( \vec{b} = (0, 1, -1, 0) \) and \( \vec{c} = (0, 0, 1, -1) \) all belong to \( \ker f \).

3. Check that the vector \( \vec{d} = (5, 5, 5, 10) \) belongs to \( \text{Im} f \).

Correction 2. 1. One has to verify that, for any vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) in \( \mathbb{R}^4 \) and any \( \lambda \in \mathbb{R} \), one has \( f(\vec{v}_1 + \lambda \vec{v}_2) = f(\vec{v}_1) + \lambda f(\vec{v}_2) \). Denote by \( (x_1, y_1, z_1, t_1) \) (resp. \( (x_2, y_2, z_2, t_2) \)) the coordinates of the vector \( \vec{v}_1 \) (resp. \( \vec{v}_2 \)) in the canonical basis of \( \mathbb{R}^4 \). The coordinates of the vector \( \vec{v}_1 + \lambda \vec{v}_2 \) are \((x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2)\). Therefore, using the formula that defines the map \( f \), one has:
\[
f(\vec{v}_1 + \lambda \vec{v}_2) = f(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2)
\]
\[
= \begin{pmatrix}
x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\
x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\
x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\
2(x_1 + \lambda x_2) + 2(y_1 + \lambda y_2) + 2(z_1 + \lambda z_2) + 2(t_1 + \lambda t_2)
\end{pmatrix}.
\]
On the other hand,

\[ f(v_1) = \begin{pmatrix} x_1 + y_1 + z_1 + t_1 \\ x_1 + y_1 + z_1 + t_1 \\ x_1 + y_1 + z_1 + t_1 \\ 2x_1 + 2y_1 + 2z_1 + 2t_1 \end{pmatrix}; \quad f(v_2) = \begin{pmatrix} x_2 + y_2 + z_2 + t_2 \\ x_2 + y_2 + z_2 + t_2 \\ x_2 + y_2 + z_2 + t_2 \\ 2x_2 + 2y_2 + 2z_2 + 2t_2 \end{pmatrix}; \]

\[ \lambda f(v_2) = \begin{pmatrix} \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(x_2 + y_2 + z_2 + t_2) \\ \lambda(2x_2 + 2y_2 + 2z_2 + 2t_2) \end{pmatrix}; \]

and

\[ f(v_1) + \lambda f(v_2) = \begin{pmatrix} x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ x_1 + y_1 + z_1 + t_1 + \lambda(x_2 + y_2 + z_2 + t_2) \\ 2x_2 + 2y_2 + 2z_2 + 2t_2 + \lambda(2x_2 + 2y_2 + 2z_2 + 2t_2) \end{pmatrix}. \]

By commutativity of the reals, one obtains \( f(v_1) + \lambda f(v_2) = f(v_1) + \lambda f(v_2). \)

The matrix of \( f \) in the canonical basis of \( \mathbb{R}^4 \) is

\[ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}. \]

2. Let us compute the images of the vectors \( \vec{a} = (1, -1, 0, 0), \vec{b} = (0, 1, -1, 0) \) and \( \vec{c} = (0, 0, 1, -1) \). One has

\[ f(\vec{a}) = f(1, -1, 0, 0) = (1 - 1, 1 - 1, 1 - 1, 2 - 2) = (0, 0, 0, 0); \]
\[ f(\vec{b}) = f(0, 1, -1, 0) = (1 - 1, 1 - 1, 1 - 2 - 2) = (0, 0, 0, 0); \]
\[ f(\vec{c}) = f(0, 0, 1, -1) = (1 - 1, 1 - 1, 1 - 2 - 2) = (0, 0, 0, 0). \]

Therefore \( \vec{a}, \vec{b} \) and \( \vec{c} \) belong to \( \ker f \).

3. Since the vector \( \vec{d} = (5, 5, 5, 10) \) is the image of the vector \( (5, 0, 0, 0) \), \( \vec{d} \) belongs to \( \text{Im} f \).

**Exercise 3.** Consider the map \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) given by:

\[ f(x, y, z) = (x + 2y + z, 2x + y + 3z, -x - y - z). \]
1. Justify that $f$ is linear.

2. Give the matrix of $f$ in the canonical basis of $\mathbb{R}^3$.

3. (a) Determine a basis and the dimension of the kernel of $f$, denoted by $\ker f$.
   
   (b) Is the map $f$ injective?

4. (a) Give the rank of $f$ and a basis of $\text{Im} f$.
   
   (b) Is the map $f$ surjective?

**Correction 3.**  
1. One has to verify that, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ and any $\lambda \in \mathbb{R}$, one has $f(\vec{v}_1 + \lambda \vec{v}_2) = f(\vec{v}_1) + \lambda f(\vec{v}_2)$. It is the same kind of computation as in Exercise 2, question 1.

2. The matrix of $f$ in the canonical basis of $\mathbb{R}^3$ is

\[
\begin{pmatrix}
1 & 2 & 1 \\
2 & 1 & 3 \\
-1 & -1 & -1
\end{pmatrix}.
\]

3. (a) The kernel of $f$, written $\ker f$, is the set of vectors which are mapped onto $\vec{0}$ by $f$. Therefore, a vector $\vec{v} = (x, y, z) \in \mathbb{R}^3$ belongs to $\ker f$ if and only if $(x, y, z)$ is a solution of the following system:

\[
\begin{align*}
x + 2y + z &= 0 \\
2x + y + 3z &= 0 \\
-x - y - z &= 0
\end{align*}
\]

Applying the Gauss method, one obtains that the above system is equivalent to

\[
\begin{align*}
x + 2y + z &= 0 \\
-3y + 2z &= 0 \\
-3y - 2z &= 0
\end{align*} \iff \begin{align*}
x + 2y + z &= 0 \\
-3y + 2z &= 0 \\
4z &= 0
\end{align*}.
\]

Therefore the unique solution of the system is $\vec{v} = \vec{0}$, and $\ker f = \{\vec{0}\}$. The dimension of $\ker f$ is therefore 0. The empty set $\emptyset$ is a basis of $\ker f$.

(b) For a linear map, being injective is equivalent to $\ker f = \{\vec{0}\}$. Hence, by the previous question, $f$ is injective.
There are many ways to answer this question. Recall that a vector \( \vec{b} \in \mathbb{R}^3 \) belongs to \( \text{Im} f \) if and only if there exists \( \vec{v} = (x, y, z) \in \mathbb{R}^3 \) such that \( f(\vec{v}) = \vec{b} \), or equivalently if \( \vec{b} \) is a linear combination of the columns of the matrix associated to \( f \). According to the expression of the matrix associated to \( f \) given in question 2., \( \text{Im} f \) is the vector space generated by the vectors 

\[
C_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.
\]

To find a basis of \( \text{Im} f \), one way is to apply Gauss algorithm to the matrix of \( f \) in order to trigonalize it. One finds:

\[
\begin{pmatrix} C_1 & C_2 & C_3 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} C_2 & -2C_1 & C_3 - C_1 \\ 1 & 0 & 0 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} C_3 & 3C_3 + C_2 \\ 1 & 0 & 0 \\ 2 & -3 & 0 \\ -1 & 1 & 1 \end{pmatrix}.
\]

Since the vectors \( \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) are column vectors of a triangular matrix, they are linearly independent. Since we applied the Gauss algorithm to the columns of the matrix associated to \( f \), they generate \( \text{Im} f \). Consequently they form a basis of \( \text{Im} f \) which is therefore equal to \( \mathbb{R}^3 \). Another way to find a basis of \( \text{Im} f \), is to compute the determinant of the matrix associated to \( f \). Since

\[
\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \neq 0,
\]

the columns of this matrix are linearly independent. Therefore they form a basis of \( \text{Im} f \).

A shorter way to answer this question, is to use Rank Theorem. Since \( f \) is an injective map from \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \), one has

\[
\dim \mathbb{R}^3 = \dim \ker f + \dim \text{Im} f \Leftrightarrow 3 = 0 + \dim \text{Im} f.
\]

Hence \( \text{Im} f = \mathbb{R}^3 \) since the only subspace of dimension 3 of \( \mathbb{R}^3 \) is \( \mathbb{R}^3 \) itself. One concludes that the rank of \( f \) (which is by definition the
dimension of $\text{Im}f$) is 3, and a basis of $\text{Im}f$ is given, for example, by $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(b) Recall that a map $f : E \to F$ is surjective if and only if $\text{Im}f = F$. For a linear map, this is equivalent to $\dim \text{Im}f = \dim F$. By the previous question, the map considered in this exercise is surjective.

Exercise 4. 1. Let $f$ be a surjective linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$. What is the dimension of the kernel of $f$?

2. Let $g$ be an injective map from $\mathbb{R}^{26}$ to $\mathbb{R}^{100}$. What is the dimension of the image of $g$?

3. Can there be a bijective linear map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$?

Correction 4. 1. By the Rank Theorem, $\dim \ker f = \dim \mathbb{R}^4 - \dim \text{Im} f$. Since $f$ is supposed to be surjective, $\dim \text{Im} f = 2$. Therefore $\dim \ker f = 4 - 2 = 2$.

2. By the Rank Theorem, $\dim \text{Im} g = \dim \mathbb{R}^{26} - \dim \ker g$. Since $g$ is supposed to be injective, $\dim \ker g = 0$. Hence $\dim \text{Im} g = 26$.

3. By the Rank Theorem, an injective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies $\dim \text{Im} f = 50$. On the other hand, a surjective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies $\dim \text{Im} f = 72$. Consequently a map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ can not be injective and surjective. Therefore there exists no bijective maps from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$.

Exercise 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix}.$$  

1. Compute a basis of the kernel of $A$.

2. Compute a basis of the image of $A$.

Correction 5. We will answer both questions at the same time. To do so, we will apply the Gauss algorithm to the columns of the matrix $A$ and $I_3$. 

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simultaneously (here $I_3$ denotes the identity matrix of size $(3, 3)$ having the same number of columns as $A$).

$$
A = \begin{pmatrix}
2 & 7 & 1 \\
-1 & 2 & 0 \\
3 & 5 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
2 & 0 & 0 \\
-1 & 11 & 1 \\
3 & -11 & -1
\end{pmatrix} \rightarrow
\begin{pmatrix}
2 & 0 & 0 \\
-1 & 11 & 0 \\
3 & -11 & 0
\end{pmatrix}.
$$

$$
I_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

It follows that a basis of $\text{Im } A$ is given by the vectors $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 11 \\ -11 \end{pmatrix}$. Indeed, the columns of the upper matrix still generate $\text{Im } A$ since we obtained them by applying the Gauss algorithm to the columns of $A$. The third column of the upper matrix being equal to the null vector, we only consider the first two columns, namely the vectors $\vec{v}_1$ and $\vec{v}_2$. These two vectors are linearly independent since they are two columns of a triangular matrix.

On the other hand the kernel of $A$ is generated by the vector $\vec{u} = \begin{pmatrix} -4 \\ -2 \\ 22 \end{pmatrix}$.

Indeed, by the Rank Theorem $\dim \ker f = \dim \mathbb{R}^3 - \dim \text{Im } f = 1$ since $\dim \text{Im } f = 2$. Moreover, one can verify that $\vec{u}$ is a non-zero vector of $\ker f$ by:

$$
A \vec{u} = \begin{pmatrix}
2 & 7 & 1 \\
-1 & 2 & 0 \\
3 & 5 & 1
\end{pmatrix} \begin{pmatrix}
-4 \\
-2 \\
22
\end{pmatrix} = \begin{pmatrix}
-8 - 14 + 22 = 0 \\
4 - 4 = 0 \\
-12 - 10 + 22 = 0
\end{pmatrix}.
$$

**Exercise 6.** Consider the matrix

$$
B = \begin{pmatrix}
1 & 2 & 3 & 1 \\
-1 & 2 & -1 & -3 \\
-3 & 5 & 2 & -3
\end{pmatrix}.
$$

1. Compute a basis of the kernel of $B$.

2. Compute a basis of the image of $B$. 

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Correction 6. We use the same technique as in the previous exercise: the Gauss algorithm on the columns of the matrix \( B \) and \( I_4 \) simultaneously (here \( I_4 \) denotes the identity matrix having as many columns as \( B \), namely 4 columns).

\[
B = \begin{pmatrix}
  1 & 2 & 3 & 1 \\
-1 & 2 & -1 & -3 \\
-3 & 5 & 2 & -3 \\
\end{pmatrix}
\]

\[
I_4 = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\begin{align*}
C_2 &\leftarrow C_2 - 2C_1 \\
C_3 &\leftarrow C_3 - 3C_1 \\
C_4 &\leftarrow C_4 - C_1
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
-1 & 4 & 2 & -2 \\
-3 & 11 & 11 & 0 \\
1 & -2 & -3 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\begin{align*}
C_3 &\leftarrow 2C_3 - C_2 \\
C_4 &\leftarrow 2C_4 + C_2
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
-1 & 4 & 0 & 0 \\
-3 & 11 & 11 & 11 \\
1 & -2 & -4 & -4 \\
0 & 1 & -1 & 1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \\
\end{pmatrix}
\]

Consequently, a basis of \( \text{Im } f \) is given by the three vectors

\[\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix},\]

\[\vec{v}_2 = \begin{pmatrix} 0 \\ 4 \\ 11 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 11 \end{pmatrix} \quad \text{. A basis of } \ker f \text{ is given by the vector}
\]

\[\vec{u} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 2 \end{pmatrix} \quad \text{.}
\]

Exercise 7. Consider the matrix

\[
C = \begin{pmatrix}
  -1 & 3 & 1 \\
  1 & 2 & 0 \\
  2 & -1 & -1 \\
  2 & 4 & 0 \\
  1 & 7 & 1
\end{pmatrix}
\]

1. Compute a basis of the kernel of \( C \).

2. Compute a basis of the image of \( C \).
Correction 7. One has

\[
C = \begin{pmatrix}
-1 & 3 & 1 \\
1 & 2 & 0 \\
2 & -1 & -1 \\
2 & 4 & 0 \\
1 & 7 & 1
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
-1 & 0 & 0 \\
1 & 5 & 1 \\
2 & 5 & 1 \\
2 & 10 & 2 \\
1 & 10 & 2
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
-1 & 0 & 0 \\
1 & 5 & 0 \\
2 & 5 & 0 \\
2 & 10 & 0 \\
1 & 10 & 0
\end{pmatrix}.
\]

Consequently a basis of $\text{Im } f$ is given by the two vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 5 \\ 10 \end{pmatrix}$. A basis of $\text{ker } f$ is given by the vector $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.