Exo7

Année 2009

Exercices de mathématiques

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Linear maps – subvectorspaces of $\mathbb{R}^n$

Exercise 1.

1. Endow $\mathbb{R}^2$ with an orthonormal frame $(O,\vec{i},\vec{j})$. Show that a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is uniquely determined by its values on the vectors $\vec{i}$ and $\vec{j}$.
2. In the basis $\{\vec{i},\vec{j}\}$, what is the matrix of the orthogonal symmetry with respect to the horizontal axis?
3. In the basis $\{\vec{i},\vec{j}\}$, what is the matrix of the orthogonal projection to the horizontal axis?
4. In the basis $\{\vec{i},\vec{j}\}$, what is the matrix of the rotation of angle θ and center O?
5. In the basis $\{\vec{i},\vec{j}\}$, what is the matrix of the homothety of center O and ratio k?
6. In the basis $\{\vec{i},\vec{j}\}$, what is the matrix of the symmetry of center O?
7. Is a translation a linear map?

Correction 1.

1. Let $f\ :\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Consider any vector $\vec{v}$ in $\mathbb{R}^2$. Since $\{\vec{i},\vec{j}\}$ is a basis of $\mathbb{R}^2$, $\vec{v}$ can be uniquely written as : $\vec{v} = x\vec{i}+y\vec{j}$. By linearity of f, one has : $f(\vec{v}) = f(x\vec{i}+y\vec{j}) = xf(\vec{i})+yf(\vec{j})$. Therefore the values of f on the vectors $\vec{i}$ and $\vec{j}$, determine the value of f on any vector of $\mathbb{R}^2$. Two linear maps taking the same values on $\vec{i}$ and $\vec{j}$ will coincide on $\mathbb{R}^2$.
2. In the basis $\{\vec{i},\vec{j}\}$, the matrix of the orthogonal symmetry with respect to the horizontal axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
3. In the basis $\{\vec{i},\vec{j}\}$, the matrix of the orthogonal projection to the horizontal axis is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

4. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the rotation of angle θ and center O is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

5. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the homothety of center O and ratio k is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

6. In the basis $\{\vec{i}, \vec{j}\}$, the matrix of the symmetry of center O is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

7. A linear map f from $\mathbb{R}^n$ into $\mathbb{R}^p$ necessarily maps $\vec{0} \in \mathbb{R}^n$ onto $\vec{0} \in \mathbb{R}^p$. The translation by a given vector $\vec{u} \in \mathbb{R}^2$ takes $\vec{v} \in \mathbb{R}^2$ to $\vec{v} + \vec{u} \in \mathbb{R}^2$. In particular, the translation of vector $\vec{u}$ takes $\vec{0} \in \mathbb{R}^2$ to $\vec{u} \in \mathbb{R}^2$. Therefore, if $\vec{u} \neq \vec{0}$, the translation of vector $\vec{u}$ is not a linear map.

Exercise 2. Let f be the map from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by:

$$f(x, y, z, t) = (x + y + z + t, x + y + z + t, x + y + z + t, 2x + 2y + 2z + 2t)\,.$$

1. Show that f is linear and write down its matrix in the canonical basis of $\mathbb{R}^4$.

2. Check that the vectors $\vec{a} = (1, -1, 0, 0)$, $\vec{b} = (0, 1, -1, 0)$ and $\vec{c} = (0, 0, 1, -1)$ all belong to $\ker f$.

3. Check that the vector $\vec{d} = (5, 5, 5, 10)$ belongs to $\mathrm{Im} f$.

Correction 2. 1. One has to verify that, for any vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^4$ and any $\lambda \in \mathbb{R}$, one has $f(\vec{v_1} + \lambda\vec{v_2}) = f(\vec{v_1}) + \lambda f(\vec{v_2})$. Denote by (x_1, y_1, z_1, t_1) (resp. (x_2, y_2, z_2, t_2)) the coordinates of the vector $\vec{v_1}$ (resp. $\vec{v_2}$) in the canonical basis of $\mathbb{R}^4$. The coordinates of the vector $\vec{v_1} + \lambda\vec{v_2}$ are $(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2)$. Therefore, using the formula that defines the map f, one has:

$$\begin{aligned} f(\vec{v_1} + \lambda\vec{v_2}) &= f(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2, t_1 + \lambda t_2) \\ &= \begin{pmatrix} x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ x_1 + \lambda x_2 + y_1 + \lambda y_2 + z_1 + \lambda z_2 + t_1 + \lambda t_2 \\ 2(x_1 + \lambda x_2) + 2(y_1 + \lambda y_2) + 2(z_1 + \lambda z_2) + 2(t_1 + \lambda t_2) \end{pmatrix}. \end{aligned}$$

On the other hand,

$$f(\vec{v_1}) = \begin{pmatrix} x_1+y_1+z_1+t_1 \\ x_1+y_1+z_1+t_1 \\ x_1+y_1+z_1+t_1 \\ 2x_1+2y_1+2z_1+2t_1 \end{pmatrix}; \quad f(\vec{v_2}) = \begin{pmatrix} x_2+y_2+z_2+t_2 \\ x_2+y_2+z_2+t_2 \\ x_2+y_2+z_2+t_2 \\ 2x_2+2y_2+2z_2+2t_2 \end{pmatrix};$$

$$\lambda f(\vec{v_2}) = \begin{pmatrix} \lambda(x_2+y_2+z_2+t_2) \\ \lambda(x_2+y_2+z_2+t_2) \\ \lambda(x_2+y_2+z_2+t_2) \\ \lambda(2x_2+2y_2+2z_2+2t_2) \end{pmatrix};$$

and

$$f(\vec{v_1})+\lambda f(\vec{v_2}) = \begin{pmatrix} x_1+y_1+z_1+t_1+\lambda(x_2+y_2+z_2+t_2) \\ x_1+y_1+z_1+t_1+\lambda(x_2+y_2+z_2+t_2) \\ x_1+y_1+z_1+t_1+\lambda(x_2+y_2+z_2+t_2) \\ 2x_2+2y_2+2z_2+2t_2+\lambda(2x_2+2y_2+2z_2+2t_2) \end{pmatrix}.$$

By commutativity of the reals, one obtains $f(\vec{v_1}+\lambda\vec{v_2}) = f(\vec{v_1})+\lambda f(\vec{v_2})$.
The matrix of f in the canonical basis of $\mathbb{R}^4$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

2. Let us compute the images of the vectors $\vec{a} = (1,-1,0,0)$, $\vec{b} = (0,1,-1,0)$ and $\vec{c} = (0,0,1,-1)$. One has

$$f(\vec{a}) = f(1,-1,0,0) = (1-1, 1-1, 1-1, 2-2) = (0,0,0,0);$$

$$f(\vec{b}) = f(0,1,-1,0) = (1-1, 1-1, 1-1, 2-2) = (0,0,0,0);$$

$$f(\vec{c}) = f(0,0,1,-1) = (1-1, 1-1, 1-1, 2-2) = (0,0,0,0).$$

Therefore $\vec{a}$, $\vec{b}$ and $\vec{c}$ belong to $\ker f$.

3. Since the vector $\vec{d} = (5,5,5,10)$ is the image of the vector $(5,0,0,0)$, $\vec{d}$ belongs to $\mathrm{Im} f$.

Exercise 3. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$f(x,y,z) = (x+2y+z, 2x+y+3z, -x-y-z).$$

1. Justify that f is linear.

2. Give the matrix of f in the canonical basis of $\mathbb{R}^3$.

3. (a) Determine a basis and the dimension of the kernel of f, denoted by $\ker f$.

   (b) Is the map f injective?

4. (a) Give the rank of f and a basis of $\text{Im} f$.

   (b) Is the map f surjective?

Correction 3. 1. One has to verify that, for any vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^3$ and any $\lambda \in \mathbb{R}$, one has $f(\vec{v_1} + \lambda\vec{v_2}) = f(\vec{v_1}) + \lambda f(\vec{v_2})$. It is the same kind of computation as in Exercise 2, question 1.

2. The matrix of f in the canonical basis of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix}.$$

3. (a) The kernel of f, written $\ker f$, is the set of vectors which are mapped onto $\vec{0}$ by f. Therefore, a vector $\vec{v} = (x, y, z) \in \mathbb{R}^3$ belongs to $\ker f$ if and only if (x, y, z) is a solution of the following system:

$$\begin{cases} x + 2y + z = 0 \\ 2x + y + 3z = 0 \\ -x - y - z = 0 \end{cases}$$

   Applying the Gauss method, one obtains that the above system is equivalent to

$$\Leftrightarrow \begin{cases} x + 2y + z = 0 \\ -3y + 2z = 0 \\ -3y - 2z = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + z = 0 \\ -3y + 2z = 0 \\ 4z = 0 \end{cases}.$$

   Therefore the unique solution of the system is $\vec{v} = \vec{0}$, and $\ker f = \{\vec{0}\}$. The dimension of $\ker f$ is therefore 0. The empty set $\emptyset$ is a basis of $\ker f$.

   (b) For a linear map, being injective is equivalent to $\ker f = \{\vec{0}\}$. Hence, by the previous question, f is injective.

4. (a) There are many ways to answer this question. Recall that a vector $\vec{b} \in \mathbb{R}^3$ belongs to Imf if and only if there exists $\vec{v} = (x, y, z) \in \mathbb{R}^3$ such that $f(\vec{v}) = \vec{b}$, or equivalently if $\vec{b}$ is a linear combination of the columns of the matrix associated to f. According to the expression of the matrix associated to f given in question 2., Imf is the vector space generated by the vectors $C_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $C_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

To find a basis of Imf, one way is to apply Gauss algorithm to the matrix of f in order to trigonalize it. One finds:

$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix} \qquad \begin{matrix} C_2 \leftarrow C_2 - 2C_1 \\ C_3 \leftarrow C_3 - C_1 \end{matrix} \qquad C_3 \leftarrow 3C_3 + C_2$$
$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Since the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are column vectors of a triangular matrix, they are linearly independent. Since we applied the Gauss algorithm to the *columns* of the matrix associated to f, they generate Imf. Consequently they form a basis of Imf which is therefore equal to $\mathbb{R}^3$.
Another way to find a basis of Imf, is to compute the determinant of the matrix associated to f. Since

$$\det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \neq 0,$$

the columns of this matrix are linearly independent. Therefore they form a basis of Imf.
A shorter way to answer this question, is to use Rank Theorem. Since f is an injective map from $\mathbb{R}^3$ into $\mathbb{R}^3$, one has

$$\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Leftrightarrow 3 = 0 + \dim \operatorname{Im} f.$$

Hence Im$f = \mathbb{R}^3$ since the only subspace of dimension 3 of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. One concludes that the rank of f (which is by definition the dimension of Imf) is 3, and a basis of Imf is given, for example, by $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(b) Recall that a map $f\ : E \to F$ is surjective if and only if Im$f = F$. For a linear map, this is equivalent to dim Imf = dim F. By the previous question, the map considered in this exercise is surjective.

Exercise 4. 1. Let f be a surjective linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$. What is the dimension of the kernel of f?

2. Let g be an injective map from $\mathbb{R}^{26}$ to $\mathbb{R}^{100}$. What is the dimension of the image of g?

3. Can there be a bijective linear map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$?

Correction 4. 1. By the Rank Theorem, dim ker f = dim $\mathbb{R}^4$−dim Im f. Since f is supposed to be surjective, dim Im$f = 2$. Therefore dim ker $f = 4 - 2 = 2$.

2. By the Rank Theorem, dim Im g = dim $\mathbb{R}^{26}$ − dim ker g. Since g is supposed to be injective, dim ker $g = 0$. Hence dim Im $g = 26$.

3. By the Rank Theorem, an injective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies dim Im $f = 50$. On the other hand, a surjective map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ satisfies dim Im $f = 72$. Consequently a map from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$ can not be injective *and* surjective. Therefore there exists no bijective maps from $\mathbb{R}^{50}$ to $\mathbb{R}^{72}$.

Exercise 5. Consider the matrix

$$A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix}.$$

1. Compute a basis of the kernel of A.

2. Compute a basis of the image of A.

Correction 5. We will answer both questions at the same time. To do so, we will apply the Gauss algorithm to the columns of the matrix A and I_3

simultanously (here I_3 denotes the identity matrix of size $(3,3)$ having the same number of columns as A).

$$A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix} \xrightarrow{\substack{C_2 \leftarrow 2C_2 - 7C_1 \\ C_3 \leftarrow 2C_3 - C_1}} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 11 & 1 \\ 3 & -11 & -1 \end{pmatrix} \xrightarrow{C_3 \leftarrow 11C_3 - C_2} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 11 & 0 \\ 3 & -11 & 0 \end{pmatrix}.$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -7 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -7 & -4 \\ 0 & 2 & -2 \\ 0 & 0 & 22 \end{pmatrix}$$

It follows that a basis of Im A is given by the vectors $\vec{v_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 0 \\ 11 \\ -11 \end{pmatrix}$. Indeed, the columns of the upper matrix still generate Im A since we obtained them by applying the Gauss algorithm to the *columns* of A. The third column of the upper matrix being equal to the null vector, we only consider the first two columns, namely the vectors $\vec{v_1}$ and $\vec{v_2}$. These two vectors are linearly independant since they are two columns of a triangular matrix.

On the other hand the kernel of A is generated by the vector $\vec{u} = \begin{pmatrix} -4 \\ -2 \\ 22 \end{pmatrix}$.

Indeed, by the Rank Theorem dim ker f = dim $\mathbb{R}^3$ − dim Im $f = 1$ since dim Im $f = 2$. Moreover, one can verify that $\vec{u}$ is a non-zero vector of ker f by:

$$A\,\vec{u} = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & 0 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ 22 \end{pmatrix} = \begin{pmatrix} -8 - 14 + 22 = 0 \\ 4 - 4 = 0 \\ -12 - 10 + 22 = 0 \end{pmatrix}.$$

Exercise 6. Consider the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -3 & 5 & 2 & -3 \end{pmatrix}.$$

1. Compute a basis of the kernel of B.
2. Compute a basis of the image of B.

Correction 6. We use the same technique as in the previous exercise: the Gauss algorithm on the columns of the matrix B and I_4 simultaneously (here I_4 denotes the identity matrix having as many columns as B, namely 4 columns).

$$B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -3 & 5 & 2 & -3 \end{pmatrix} \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{C_2 \leftarrow C_2 - 2C_1 \\ C_3 \leftarrow C_3 - 3C_1 \\ C_4 - C_1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 2 & -2 \\ -3 & 11 & 11 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\substack{C_3 \leftarrow 2C_3 - C_2 \\ C_4 \leftarrow 2C_4 + C_2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -3 & 11 & 11 & 11 \end{pmatrix} \begin{pmatrix} 1 & -2 & -4 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{C_4 \leftarrow C_4 - C_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -3 & 11 & 11 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Consequently, a basis of Im f is given by the three vectors $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$, $\vec{v_2} = \begin{pmatrix} 0 \\ 4 \\ 11 \end{pmatrix}$ and $\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$. A basis of ker f is given by the vector $\vec{u} = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 2 \end{pmatrix}$.

Exercise 7. Consider the matrix

$$C = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \\ 2 & 4 & 0 \\ 1 & 7 & 1 \end{pmatrix}.$$

1. Compute a basis of the kernel of C.

2. Compute a basis of the image of C.

Correction 7. One has

$$\begin{array}{ccccc} C = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & -1 \\ 2 & 4 & 0 \\ 1 & 7 & 1 \end{pmatrix} & \xrightarrow{\substack{C_2 \leftarrow C_2 + 3C_1 \\ C_3 \leftarrow C_3 + C_1}} & \begin{pmatrix} -1 & 0 & 0 \\ 1 & 5 & 1 \\ 2 & 5 & 1 \\ 2 & 10 & 2 \\ 1 & 10 & 2 \end{pmatrix} & \xrightarrow{C_3 \leftarrow 5C_3 - C_2} & \begin{pmatrix} -1 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & 5 & 0 \\ 2 & 10 & 0 \\ 1 & 10 & 0 \end{pmatrix} \\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{pmatrix} \end{array}.$$

Consequently a basis of Im f is given by the two vectors $\vec{v_1} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 10 \\ 10 \end{pmatrix}$. A basis of ker f is given by the vector $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.